

QED in arbitrary linear media: amplifying media

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Abstract. Recently, we have developed a unified approach to QED in arbitrary linearly responding media in equilibrium—media that give rise to absorption [Phys. Rev. A **75**, (2007) 053813]. In the present paper we show that, under appropriate conditions, the theory can be quite naturally generalized to amplifying media the effect of which is described within the framework of linear response theory. We discuss the limits of validity of the generalized theory and make contact with earlier quantization schemes suggested for the case of linearly and locally responding amplifying dielectric-type media. To illustrate the theory, we present the electromagnetic-field correlation functions that determine the Casimir force in the presence of amplifying media.

1 Introduction

In macroscopic electrodynamics, the effect of an arbitrary linearly responding medium in equilibrium can be described by means of a complex conductivity tensor of the unperturbed medium left to its own resources. It is well known that any electromagnetic excitation in such a medium eventually dissipates due to unavoidable absorption in the whole frequency range. On the other hand, if a medium has been brought out of equilibrium, it may act, in a certain frequency range, as an amplifier until after some relaxation time it will have reached the equilibrium state again. A detailed description of the temporal evolution of the medium properties during such transient processes is a highly difficult problem in general. However, when—by implementation of a suitable energy supply mechanism—a medium is pumped in such a way that an (externally controlled) quasi-stationary regime is established and maintained for a sufficiently long time, one may describe the effect of the amplifying medium by assigning to it a conductivity tensor. The properties of such an amplifying medium are thereby regarded as being fixed—an assumption which is of course only justified in a time interval within which significant changes of these properties can be ignored (for the purpose at hand). Clearly, the conductivity tensor that describes the linear response of an amplifying medium can be expected to have all the properties known from an absorbing medium, except that it is no longer the kernel of a positive-definite integral operator.

Both absorption and amplification introduce additional noise into the system—noise that must be carefully taken into account in quantum electrodynamics (QED). Recently, we have developed a unified approach to macroscopic QED in arbitrary linearly responding media described by a conductivity tensor for absorbing media [1]. In the present paper, we extend this quantization scheme to allow also for amplifying media. In Sec. 2, we outline the main aspects of the scheme and indicate at which points modifications are necessary when a medium becomes amplifying. The extension of the theory to amplifying media is then given in Sec. 3, where we also make contact with earlier quantization schemes for the electromagnetic field in

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(locally responding dielectric) amplifying media [2,3]. In this context, we also address the problem of the necessity of a modification of the Hamiltonian, which has not been considered in Refs. [2,3]. It is worth noting that without this modification the field operators, whose positive-frequency components are associated with both annihilation and creation operators in the case of amplifying media, would fail to satisfy Maxwell's equations. To illustrate the theory, and to make contact with recent work on the Casimir force in the presence of linearly amplifying media [4], we present the vacuum-field correlation functions that are relevant to the Casimir force. The example shows that, as expected, the form of the field correlation functions changes significantly, compared to the familiar form observed in the case of equilibrium media, when amplifying media are involved—a point that has not been taken into account in Ref. [4]. In Sec. 4, we discuss in detail the limits of validity of the theory. Finally, we give a summary and some concluding remarks in Sec. 5.

2 Sketch of the quantization scheme

It is well-known that Maxwell's equations imply that the electric field in the frequency domain, $\underline{\mathbf{E}}(\mathbf{r}, \omega)$, obeys, in the presence of matter, the (classical) equation

$$\nabla \times \nabla \times \underline{\mathbf{E}}(\mathbf{r}, \omega) - \frac{\omega^2}{c^2} \underline{\mathbf{E}}(\mathbf{r}, \omega) = i\mu_0\omega \underline{\mathbf{j}}(\mathbf{r}, \omega), \quad (1)$$

where $\underline{\mathbf{j}}(\mathbf{r}, \omega)$ is the total current density, i. e., the current density that covers all the matter (on a chosen length scale). If $\underline{\mathbf{j}}(\mathbf{r}, \omega)$ can be attributed entirely to a medium whose internal atomistic structure need not be resolved, we may assume, within the framework of linear response theory, the constitutive relation

$$\underline{\mathbf{j}}(\mathbf{r}, \omega) = \int d^3r' \mathbf{Q}(\mathbf{r}, \mathbf{r}', \omega) \cdot \underline{\mathbf{E}}(\mathbf{r}', \omega) + \underline{\mathbf{j}}_{\text{N}}(\mathbf{r}, \omega), \quad (2)$$

where $\mathbf{Q}(\mathbf{r}, \mathbf{r}', \omega)$ is the complex, macroscopic conductivity tensor of the unperturbed medium in the frequency domain [5,6], and $\underline{\mathbf{j}}_{\text{N}}(\mathbf{r}, \omega)$ is a Langevin noise source. Equation (2) covers all the possible features of a linear medium (in particular, temporal as well as spatial dispersion). According to Onsager's reciprocity theorem [5,6], the conductivity tensor fulfills the reciprocity relation $Q_{ij}(\mathbf{r}, \mathbf{r}', \omega) = Q_{ji}(\mathbf{r}', \mathbf{r}, \omega)$, which we will adopt throughout the paper. For chosen ω , $\mathbf{Q}(\mathbf{r}, \mathbf{r}', \omega)$ can be assumed to be the integral kernel of a reasonably well-behaved integral operator acting on vector functions in position space. In particular, in the spatially non-dispersive limit, it may become a (quasi-)local integral kernel, i. e., a linear combination of δ -functions and their derivatives. It should be noted that the decomposition of the conductivity tensor into real and imaginary parts, $\mathbf{Q}(\mathbf{r}, \mathbf{r}', \omega) = \text{Re } \mathbf{Q}(\mathbf{r}, \mathbf{r}', \omega) + i \text{Im } \mathbf{Q}(\mathbf{r}, \mathbf{r}', \omega)$, corresponds, due to the reciprocity of $\mathbf{Q}(\mathbf{r}, \mathbf{r}', \omega)$, to the decomposition of the associated operator into Hermitian and anti-Hermitian parts,

$$\boldsymbol{\sigma}(\mathbf{r}, \mathbf{r}', \omega) \equiv \text{Re } \mathbf{Q}(\mathbf{r}, \mathbf{r}', \omega) = \frac{1}{2} [\mathbf{Q}(\mathbf{r}, \mathbf{r}', \omega) + \mathbf{Q}^+(\mathbf{r}', \mathbf{r}, \omega)], \quad (3)$$

$$\boldsymbol{\tau}(\mathbf{r}, \mathbf{r}', \omega) \equiv \text{Im } \mathbf{Q}(\mathbf{r}, \mathbf{r}', \omega) = \frac{1}{2i} [\mathbf{Q}(\mathbf{r}, \mathbf{r}', \omega) - \mathbf{Q}^+(\mathbf{r}', \mathbf{r}, \omega)]. \quad (4)$$

Throughout the paper, the superscripts T and $+$ are used to indicate transposition and Hermitian conjugation with respect to tensor indices. Since $\mathbf{Q}(\mathbf{r}, \mathbf{r}', \omega)$ is the temporal Fourier transform of a response function [5,7,8], it is analytic in the upper complex ω half-plane, fulfills the Kramers–Kronig (Hilbert transform) relations, and satisfies the Schwarz reflection principle $\mathbf{Q}^*(\mathbf{r}, \mathbf{r}', \omega) = \mathbf{Q}(\mathbf{r}, \mathbf{r}', -\omega^*)$. Since an unperturbed medium left to its own resources is necessarily an absorbing one, the operator associated with the integral kernel $\boldsymbol{\sigma}(\mathbf{r}, \mathbf{r}', \omega)$ is a positive definite operator (see, e. g., Refs. [5,6]), i. e., for any (quadratically integrable) vector function $\mathbf{v}(\mathbf{r})$ the inequality

$$\int d^3r \int d^3r' \mathbf{v}^*(\mathbf{r}) \cdot \boldsymbol{\sigma}(\mathbf{r}, \mathbf{r}', \omega) \cdot \mathbf{v}(\mathbf{r}') > 0, \quad (5)$$

which is characteristic of absorbing media, must hold.

Inserting Eq.(2) in Eq. (1), we find that $\underline{\mathbf{E}}(\mathbf{r}, \omega)$ satisfies the integro-differential equation

$$\nabla \times \nabla \times \underline{\mathbf{E}}(\mathbf{r}, \omega) - \frac{\omega^2}{c^2} \underline{\mathbf{E}}(\mathbf{r}, \omega) - i\mu_0\omega \int d^3r' \mathbf{Q}(\mathbf{r}, \mathbf{r}', \omega) \cdot \underline{\mathbf{E}}(\mathbf{r}', \omega) = i\mu_0\omega \underline{\mathbf{j}}_N(\mathbf{r}, \omega), \quad (6)$$

which can be regarded as being a Langevin equation. Note that $\underline{\mathbf{j}}_N(\mathbf{r}, \omega)$, which is a classical quantity yet, has vanishing mean value (as a stochastic variable) and is related to the conductivity tensor through a fluctuation-dissipation relation. For a detailed discussion of Langevin equations, we refer the reader to, e.g., Refs. [5,9,10]. The solution to Eq. (6) can be given in the form

$$\underline{\mathbf{E}}(\mathbf{r}, \omega) = i\mu_0\omega \int d^3r' \mathbf{G}(\mathbf{r}, \mathbf{r}', \omega) \cdot \underline{\mathbf{j}}_N(\mathbf{r}', \omega), \quad (7)$$

where $\mathbf{G}(\mathbf{r}, \mathbf{r}', \omega)$ is the retarded Green tensor in the frequency domain. It satisfies Eq. (6) with the (tensorial) δ -function source,

$$\nabla \times \nabla \times \mathbf{G}(\mathbf{r}, \mathbf{s}, \omega) - \frac{\omega^2}{c^2} \mathbf{G}(\mathbf{r}, \mathbf{s}, \omega) - i\mu_0\omega \int d^3r' \mathbf{Q}(\mathbf{r}, \mathbf{r}', \omega) \cdot \mathbf{G}(\mathbf{r}', \mathbf{s}, \omega) = \mathbf{I}\delta(\mathbf{r} - \mathbf{s}) \quad (8)$$

$[\mathbf{I}$, unit tensor], together with the boundary condition at infinity, and has all the attributes of a Fourier transformed response function just as $\mathbf{Q}(\mathbf{r}, \mathbf{r}', \omega)$ has them. In particular, it is analytic in the upper complex ω half-plane and the Schwarz reflection principle $\mathbf{G}^*(\mathbf{r}, \mathbf{r}', \omega) = \mathbf{G}(\mathbf{r}, \mathbf{r}', -\omega^*)$ is valid. Since $\mathbf{Q}(\mathbf{r}, \mathbf{r}', \omega)$ is reciprocal, so is $\mathbf{G}(\mathbf{r}, \mathbf{r}', \omega)$, $\mathbf{G}(\mathbf{r}, \mathbf{r}', \omega) = \mathbf{G}^T(\mathbf{r}', \mathbf{r}, \omega)$, and, for real ω , the generalized integral relation

$$\mu_0\omega \int d^3s \int d^3s' \mathbf{G}(\mathbf{r}, \mathbf{s}, \omega) \cdot \boldsymbol{\sigma}(\mathbf{s}, \mathbf{s}', \omega) \cdot \mathbf{G}^*(\mathbf{s}', \mathbf{r}', \omega) = \text{Im } \mathbf{G}(\mathbf{r}, \mathbf{r}', \omega) \quad (9)$$

holds (see Ref. [1]). For absorbing media, Eq. (7) is the unique solution to Eq. (6), i.e., it is not to be supplemented with any solutions of the source-free version of Eq. (6).

Now, one can go over to the quantum-theoretical formulation of the theory [1]. Writing the operators of the electric and magnetic induction fields $\hat{\mathbf{E}}(\mathbf{r})$ and $\hat{\mathbf{B}}(\mathbf{r})$, respectively, in a picture-independent manner as

$$\hat{\mathbf{E}}(\mathbf{r}) = \int_0^\infty d\omega \underline{\hat{\mathbf{E}}}(\mathbf{r}, \omega) + \text{H.c.}, \quad (10)$$

$$\hat{\mathbf{B}}(\mathbf{r}) = \int_0^\infty d\omega \underline{\hat{\mathbf{B}}}(\mathbf{r}, \omega) + \text{H.c.}, \quad (11)$$

where the respective positive-frequency parts $\underline{\hat{\mathbf{E}}}(\mathbf{r}, \omega)$ and $\underline{\hat{\mathbf{B}}}(\mathbf{r}, \omega) = (i\omega)^{-1} \nabla \times \underline{\hat{\mathbf{E}}}(\mathbf{r}, \omega)$ are expressed in terms of the henceforth operator-valued noise current density as [cf. Eq. (7)]

$$\underline{\hat{\mathbf{E}}}(\mathbf{r}, \omega) = i\mu_0\omega \int d^3r' \mathbf{G}(\mathbf{r}, \mathbf{r}', \omega) \cdot \underline{\hat{\mathbf{j}}}_N(\mathbf{r}', \omega), \quad (12)$$

$$\underline{\hat{\mathbf{B}}}(\mathbf{r}, \omega) = \mu_0 \nabla \times \int d^3r' \mathbf{G}(\mathbf{r}, \mathbf{r}', \omega) \cdot \underline{\hat{\mathbf{j}}}_N(\mathbf{r}', \omega), \quad (13)$$

one can prove (Appendix A) that the well known fundamental commutation relation

$$[\hat{\mathbf{E}}(\mathbf{r}), \hat{\mathbf{B}}(\mathbf{r}')] = i\hbar \nabla \times \mathbf{I} \delta(\mathbf{r} - \mathbf{r}') / \varepsilon_0 \quad (14)$$

is satisfied if one lets

$$[\underline{\hat{\mathbf{j}}}_N(\mathbf{r}, \omega), \underline{\hat{\mathbf{j}}}_N^\dagger(\mathbf{r}', \omega')] = \frac{\hbar\omega}{\pi} \boldsymbol{\sigma}(\mathbf{r}, \mathbf{r}', \omega) \delta(\omega - \omega'). \quad (15)$$

To ensure that the temporal evolution of the electromagnetic field is in accordance with Maxwell's equations, the operators $\hat{\mathbf{j}}_{\mathbf{N}}(\mathbf{r}, \omega)$ and $\hat{\mathbf{j}}_{\mathbf{N}}^\dagger(\mathbf{r}, \omega)$ have to evolve, in the Heisenberg-picture, like $\sim e^{-i\omega t}$ and $\sim e^{i\omega t}$, respectively. Hence, a Hamiltonian \hat{H} as a functional of $\hat{\mathbf{j}}_{\mathbf{N}}(\mathbf{r}, \omega)$ and $\hat{\mathbf{j}}_{\mathbf{N}}^\dagger(\mathbf{r}, \omega)$ needs to be introduced such that

$$[\hat{\mathbf{j}}_{\mathbf{N}}(\mathbf{r}, \omega), \hat{H}] = \hbar\omega \hat{\mathbf{j}}_{\mathbf{N}}(\mathbf{r}, \omega), \quad (16)$$

which constrains the Hamiltonian to take the form

$$\hat{H} = \pi \int_0^\infty d\omega \int d^3r \int d^3r' \hat{\mathbf{j}}_{\mathbf{N}}^\dagger(\mathbf{r}, \omega) \cdot \boldsymbol{\rho}(\mathbf{r}, \mathbf{r}', \omega) \cdot \hat{\mathbf{j}}_{\mathbf{N}}(\mathbf{r}', \omega), \quad (17)$$

apart from an irrelevant c -number contribution. From Eq. (16) together with Eqs. (15) and (17) one can see that $\boldsymbol{\rho}(\mathbf{r}, \mathbf{r}', \omega)$ has to be chosen to be the kernel of the integral operator that is the inverse of the integral operator associated with $\boldsymbol{\sigma}(\mathbf{r}, \mathbf{r}', \omega)$,

$$\int d^3s \boldsymbol{\rho}(\mathbf{r}, \mathbf{s}, \omega) \cdot \boldsymbol{\sigma}(\mathbf{s}, \mathbf{r}', \omega) = \int d^3s \boldsymbol{\sigma}(\mathbf{r}, \mathbf{s}, \omega) \cdot \boldsymbol{\rho}(\mathbf{s}, \mathbf{r}', \omega) = \mathbf{I} \delta(\mathbf{r} - \mathbf{r}'). \quad (18)$$

Note that, by means of the correspondence $i(\varepsilon_0\omega)^{-1} \mathbf{Q}(\mathbf{r}, \mathbf{r}', \omega) \leftrightarrow \boldsymbol{\chi}(\mathbf{r}, \mathbf{r}', \omega)$, where $\boldsymbol{\chi}(\mathbf{r}, \mathbf{r}', \omega)$ is the (nonlocal) dielectric susceptibility tensor, the basic commutation relation (15) becomes equivalent to the commutation relation derived from a microscopic, linear two-band model of dielectric material [11], which has been used to study the quantized electromagnetic field in spatially dispersive dielectrics [12,13]. Provided the observables of interest—including the Hamiltonian—can be viewed as functionals of $\hat{\mathbf{j}}_{\mathbf{N}}(\mathbf{r}, \omega)$ [rather than functionals of the individual current contributions in a possible decomposition of $\hat{\mathbf{j}}_{\mathbf{N}}(\mathbf{r}, \omega)$, see Ref. [1]], Eqs. (15) and (17) can be regarded, in view of the fluctuation-dissipation theorem(s) (see, e. g., Ref. [5]), as being unique, and hence, as invariable fundament of the theory.

If, within some frequency interval, $\boldsymbol{\sigma}(\mathbf{r}, \mathbf{r}', \omega)$ does not correspond to a positive definite integral operator so that the integral in the inequality (5) becomes negative for suitably chosen (quadratically integrable) functions $\mathbf{v}(\mathbf{r})$, linear amplification is possible in this frequency interval. It is not difficult to prove that the equations given above can be maintained even if the positivity condition (5) is abandoned, provided that nevertheless (i) the Green tensor retains its analytic properties and (ii) $\boldsymbol{\rho}(\mathbf{r}, \mathbf{r}', \omega)$ continues to exist (for a detailed discussion of these conditions, see Sec. 4). However, the diagonal form

$$\hat{H} = \int_0^\infty d\omega \hbar\omega \int d^3r \hat{\mathbf{f}}^\dagger(\mathbf{r}, \omega) \cdot \hat{\mathbf{f}}(\mathbf{r}, \omega) \quad (19)$$

in which the Hamiltonian (17) can be brought in the case of absorbing media by means of a linear (and invertible) transformation of the variables,

$$\hat{\mathbf{j}}_{\mathbf{N}}(\mathbf{r}, \omega) = \left(\frac{\hbar\omega}{\pi} \right)^{\frac{1}{2}} \int d^3r' \mathbf{K}(\mathbf{r}, \mathbf{r}', \omega) \cdot \hat{\mathbf{f}}(\mathbf{r}', \omega), \quad (20)$$

cannot be maintained if the condition (5) is abandoned. In Eqs. (19) and (20), $\hat{\mathbf{f}}(\mathbf{r}, \omega)$ is a bosonic field,

$$[\hat{\mathbf{f}}(\mathbf{r}, \omega), \hat{\mathbf{f}}^\dagger(\mathbf{r}', \omega')] = \mathbf{I} \delta(\mathbf{r} - \mathbf{r}') \delta(\omega - \omega'), \quad (21)$$

and the integral kernel $\mathbf{K}(\mathbf{r}, \mathbf{r}', \omega)$ has to obey, for real ω , the integral equation

$$\int d^3s \mathbf{K}(\mathbf{r}, \mathbf{s}, \omega) \cdot \mathbf{K}^+(\mathbf{r}', \mathbf{s}, \omega) = \boldsymbol{\sigma}(\mathbf{r}, \mathbf{r}', \omega) \quad (22)$$

for Eq. (19) to be an equivalent representation of Eq. (17). In the case of absorbing media, one may evidently construct a possible integral kernel $\mathbf{K}(\mathbf{r}, \mathbf{r}', \omega)$ in the form of

$$\mathbf{K}(\mathbf{r}, \mathbf{r}', \omega) = \int d\alpha \sigma^{\frac{1}{2}}(\alpha, \omega) \mathbf{F}(\alpha, \mathbf{r}, \omega) \mathbf{F}^*(\alpha, \mathbf{r}', \omega) \quad (23)$$

$[\sigma^{1/2}(\alpha, \omega) > 0]$, where the complete and (δ) -orthonormal functions $\mathbf{F}(\alpha, \mathbf{r}, \omega)$,

$$\int d\alpha \mathbf{F}(\alpha, \mathbf{r}, \omega) \mathbf{F}^*(\alpha, \mathbf{r}', \omega) = \mathbf{I} \delta(\mathbf{r} - \mathbf{r}'), \quad (24)$$

$$\int d^3r \mathbf{F}^*(\alpha, \mathbf{r}, \omega) \cdot \mathbf{F}(\alpha', \mathbf{r}, \omega) = \delta(\alpha - \alpha'), \quad (25)$$

are defined by the eigenvalue problem

$$\int d^3r' \boldsymbol{\sigma}(\mathbf{r}, \mathbf{r}', \omega) \cdot \mathbf{F}(\alpha, \mathbf{r}', \omega) = \sigma(\alpha, \omega) \mathbf{F}(\alpha, \mathbf{r}, \omega). \quad (26)$$

Here, the real ω plays the role of a parameter and α stands for the set of discrete and/or continuous quantities needed to label the eigenfunctions. (An α -integration therefore symbolizes multiple summations and/or integrations over all those quantities.) Note that the kernel (23) is not unique but is determined by Eq. (22) only up to a unitary transformation, which corresponds to the possibility of redefining the dynamical variables $\hat{\mathbf{f}}(\mathbf{r}, \omega)$; for details, see Ref. [1]. Equation (22) is inconsistent if $\boldsymbol{\sigma}(\mathbf{r}, \mathbf{r}', \omega)$ is not the kernel of a positive definite integral operator so that in this case no valid kernel $\mathbf{K}(\mathbf{r}, \mathbf{r}', \omega)$ exists at all. Hence Eq. (19) would fail, if $\hat{\mathbf{f}}(\mathbf{r}, \omega)$ were regarded as being a bosonic field satisfying the commutation relation (21).

3 Extension to linearly amplifying media

In order to include in the quantization scheme linearly amplifying media, we first note that, although Eq. (22) cannot be satisfied anymore so that Eqs. (19)–(21) do not apply either, Eqs. (24)–(26) remain valid, with $\sigma(\alpha, \omega)$ not being restricted to positive values anymore. We may therefore expand $\boldsymbol{\rho}(\mathbf{r}, \mathbf{r}', \omega)$ as

$$\boldsymbol{\rho}(\mathbf{r}, \mathbf{r}', \omega) = \int d\alpha \sigma^{-1}(\alpha, \omega) \mathbf{F}(\alpha, \mathbf{r}, \omega) \mathbf{F}^*(\alpha, \mathbf{r}', \omega), \quad (27)$$

which enables us to rewrite the Hamiltonian (17) as

$$\hat{H} = \int_0^\infty d\omega \hbar \omega \int d\alpha \operatorname{sgn} \sigma(\alpha, \omega) \hat{g}^\dagger(\alpha, \omega) \hat{g}(\alpha, \omega), \quad (28)$$

where

$$\hat{g}(\alpha, \omega) = \left(\frac{\hbar \omega}{\pi} \right)^{-\frac{1}{2}} |\sigma(\alpha, \omega)|^{-\frac{1}{2}} \int d^3r \mathbf{F}^*(\alpha, \mathbf{r}, \omega) \cdot \hat{\mathbf{j}}_{\mathbf{N}}(\mathbf{r}, \omega). \quad (29)$$

With the help of Eqs. (15), (25), and (26), it is not difficult to see that

$$[\hat{g}(\alpha, \omega), \hat{g}^\dagger(\alpha', \omega')] = \operatorname{sgn} \sigma(\alpha, \omega) \delta(\alpha - \alpha') \delta(\omega - \omega'). \quad (30)$$

The $\hat{g}(\alpha, \omega)$ may be viewed as non-bosonic generalizations of the natural variables considered in Ref. [1], as they coincide with the latter in the purely absorbing case where $\operatorname{sgn} \sigma(\alpha, \omega) = 1$. Inversion of Eq. (29) by means of Eq. (24) yields

$$\hat{\mathbf{j}}_{\mathbf{N}}(\mathbf{r}, \omega) = \left(\frac{\hbar \omega}{\pi} \right)^{\frac{1}{2}} \int d\alpha |\sigma(\alpha, \omega)|^{\frac{1}{2}} \mathbf{F}(\alpha, \mathbf{r}, \omega) \hat{g}(\alpha, \omega). \quad (31)$$

The commutation relation (30) shows that $\hat{g}(\alpha, \omega)$ [$\hat{g}^\dagger(\alpha, \omega)$] is a bosonic annihilation (creation) operator for positive eigenvalues $\sigma(\alpha, \omega)$ whereas for negative ones, $\hat{g}(\alpha, \omega)$ [$\hat{g}^\dagger(\alpha, \omega)$] is a creation (annihilation) operator. It makes therefore sense to rename the operators according

to this behavior. Thus, denoting in each of the two cases $\text{sgn } \sigma(\alpha, \omega) = \pm 1$ the respective annihilation operator by $\hat{b}(\alpha, \omega)$ and the respective creation operator by $\hat{b}^\dagger(\alpha, \omega)$, we can rewrite Eqs. (28), (30), and (31) as

$$\hat{H} = \int_0^\infty d\omega \hbar\omega \int_{(+)} d\alpha \hat{b}^\dagger(\alpha, \omega) \hat{b}(\alpha, \omega) - \int_0^\infty d\omega \hbar\omega \int_{(-)} d\alpha \hat{b}(\alpha, \omega) \hat{b}^\dagger(\alpha, \omega), \quad (32)$$

$$[\hat{b}(\alpha, \omega), \hat{b}^\dagger(\alpha', \omega')] = \delta(\alpha - \alpha') \delta(\omega - \omega'), \quad (33)$$

and

$$\hat{\mathbf{j}}_{\text{N}}(\mathbf{r}, \omega) = \left(\frac{\hbar\omega}{\pi} \right)^{\frac{1}{2}} \left\{ \int_{(+)} d\alpha \sigma^{\frac{1}{2}}(\alpha, \omega) \mathbf{F}(\alpha, \mathbf{r}, \omega) \hat{b}(\alpha, \omega) + \int_{(-)} d\alpha [-\sigma(\alpha, \omega)]^{\frac{1}{2}} \mathbf{F}(\alpha, \mathbf{r}, \omega) \hat{b}^\dagger(\alpha, \omega) \right\}, \quad (34)$$

respectively, where the notation $\int_{(\pm)} d\alpha \cdots$ means that the integration extends over those values for which $\text{sgn } \sigma(\alpha, \omega) = \pm 1$. Note that the ranges of integration depend on the chosen frequency in general. It may be convenient to change to normal order the second term in Eq. (32), $\hat{b}(\alpha, \omega) \hat{b}^\dagger(\alpha, \omega) \mapsto : \hat{b}(\alpha, \omega) \hat{b}^\dagger(\alpha, \omega) := \hat{b}^\dagger(\alpha, \omega) \hat{b}(\alpha, \omega)$, i. e., to replace the Hamiltonian in Eq. (32) with

$$\hat{H} = \int_0^\infty d\omega \hbar\omega \int d\alpha \text{sgn } \sigma(\alpha, \omega) \hat{b}^\dagger(\alpha, \omega) \hat{b}(\alpha, \omega), \quad (35)$$

which differs from the Hamiltonian in Eq. (32) by an (infinite but) irrelevant c -number. Since the state space of the system is to be constructed by means of the bosonic variables $\hat{b}(\alpha, \omega)$ and $\hat{b}^\dagger(\alpha, \omega)$, the use of Eq. (35) in place of Eq. (32) is equivalent to a redefinition (renormalization) of the zero of energy by the condition of absence of quanta (vacuum state). It should be stressed that the temporal evolution of the variables $\hat{b}(\alpha, \omega)$ and $\hat{b}^\dagger(\alpha, \omega)$ which follows from Eq. (35) [together with Eq. (33)] is sensitive to the sign of $\sigma(\alpha, \omega)$ in just such a way that Eq. (34) always represents the positive-frequency part of the noise current density, as required.

From Eq. (35) it is seen that there is a continuum of negative energy eigenvalues in the case of amplification, in addition to the positive-energy continuum associated with absorption. Thus, the vacuum state can no longer be said to be the ground state of the system—there is in fact no ground state as the continuum stretches down to $-\infty$. This somewhat unpleasant feature is due to the fact that the pump mechanism which prepares the medium to act, in some frequency interval, as an amplifier is not dynamically included in the theory. Clearly, the approximation to treat the effect of pumping within the framework of linear response theory breaks down when states with large negative energies are significantly involved.

By means of the transformation

$$\begin{aligned} \hat{\mathbf{f}}(\mathbf{r}, \omega) &= \int d\alpha \mathbf{F}(\alpha, \mathbf{r}, \omega) \hat{g}(\alpha, \omega) \\ &= \int_{(+)} d\alpha \mathbf{F}(\alpha, \mathbf{r}, \omega) \hat{b}(\alpha, \omega) + \int_{(-)} d\alpha \mathbf{F}(\alpha, \mathbf{r}, \omega) \hat{b}^\dagger(\alpha, \omega), \end{aligned} \quad (36)$$

vectorial field variables $\hat{\mathbf{f}}(\mathbf{r}, \omega)$ can be introduced, which can be viewed as generalizations of the variables $\hat{\mathbf{f}}(\mathbf{r}, \omega)$ introduced in Eq. (20) for the case of purely absorbing media. Employing Eq. (25) to invert (the first equation in) Eq. (36) and inserting the result in Eq. (31), we obtain the generalization of Eq. (20) [$\hat{\mathbf{f}}(\mathbf{r}, \omega) \mapsto \hat{\mathbf{f}}(\mathbf{r}, \omega)$], from which we can read off, by comparison with Eq. (20), the generalization of the kernel $\mathbf{K}(\mathbf{r}, \mathbf{r}', \omega)$, in the form of its eigenfunction expansion. Not surprisingly, it is given by Eq. (23) with $\sigma(\alpha, \omega)$ being replaced by $|\sigma(\alpha, \omega)|$. (It is of course possible to consider equivalent kernels just as in the case of purely absorbing

media.) Unfortunately, the variables $\hat{\mathbf{f}}(\mathbf{r}, \omega)$ do not diagonalize the Hamiltonian in general, and are thus less useful than in the case of purely absorbing media. From Eqs. (30) and (36), it follows that

$$[\hat{\mathbf{f}}(\mathbf{r}, \omega), \hat{\mathbf{f}}^\dagger(\mathbf{r}', \omega')] = \delta(\omega - \omega') \int d\alpha \operatorname{sgn} \sigma(\alpha, \omega) \mathbf{F}(\alpha, \mathbf{r}, \omega) \mathbf{F}^*(\alpha, \mathbf{r}', \omega). \quad (37)$$

The integral appearing on the right-hand side is the kernel of a parity-type operator, which reduces to the unit operator in the case of purely absorbing media, as it should be.

A noteworthy simplification occurs if the medium response is strictly local—an assumption that is typically made in the study of amplifying media. In this case, the $\hat{\mathbf{f}}(\mathbf{r}, \omega)$ are related to the $\hat{g}(\alpha, \omega)$ [$\alpha \mapsto (i, \mathbf{r})$] in a very simple way just as in the case of purely absorbing media that respond locally [since the eigenfunctions $\mathbf{F}(\alpha, \mathbf{r}, \omega)$ are spatially localized in this case, see Ref. [1]]. Focusing, for simplicity, on isotropic media, we may then rewrite Eqs. (31) and (28), respectively, as

$$\hat{\mathbf{j}}_{\mathbf{N}}(\mathbf{r}, \omega) = \left(\frac{\hbar\omega}{\pi} \right)^{\frac{1}{2}} |\sigma(\mathbf{r}, \omega)|^{\frac{1}{2}} \hat{\mathbf{f}}(\mathbf{r}, \omega), \quad (38)$$

and

$$\hat{H} = \int_0^\infty d\omega \hbar\omega \int d^3r \operatorname{sgn} \sigma(\mathbf{r}, \omega) \hat{\mathbf{f}}^\dagger(\mathbf{r}, \omega) \cdot \hat{\mathbf{f}}(\mathbf{r}, \omega), \quad (39)$$

and (37) simplifies to

$$[\hat{\mathbf{f}}(\mathbf{r}, \omega), \hat{\mathbf{f}}^\dagger(\mathbf{r}', \omega')] = \operatorname{sgn} \sigma(\mathbf{r}, \omega) \mathbf{I} \delta(\mathbf{r} - \mathbf{r}') \delta(\omega - \omega'). \quad (40)$$

Now we can switch to genuine bosonic variables by renaming $\hat{\mathbf{f}}(\mathbf{r}, \omega)$ as $\hat{\mathbf{b}}(\mathbf{r}, \omega)$ and $\hat{\mathbf{b}}^\dagger(\mathbf{r}, \omega)$ for $\operatorname{sgn} \sigma(\mathbf{r}, \omega) = 1$ and $\operatorname{sgn} \sigma(\mathbf{r}, \omega) = -1$, respectively, so that Eq. (40) changes to

$$[\hat{\mathbf{b}}(\mathbf{r}, \omega), \hat{\mathbf{b}}^\dagger(\mathbf{r}', \omega')] = \mathbf{I} \delta(\mathbf{r} - \mathbf{r}') \delta(\omega - \omega'). \quad (41)$$

The noise current density in Eq. (34) and the Hamiltonian in Eq. (35), respectively, can then be expressed in terms of $\hat{\mathbf{b}}(\mathbf{r}, \omega)$ and $\hat{\mathbf{b}}^\dagger(\mathbf{r}, \omega)$ as

$$\hat{\mathbf{j}}_{\mathbf{N}}(\mathbf{r}, \omega) = \left(\frac{\hbar\omega}{\pi} \right)^{\frac{1}{2}} \left\{ \theta[\sigma(\mathbf{r}, \omega)] \sigma^{\frac{1}{2}}(\mathbf{r}, \omega) \hat{\mathbf{b}}(\mathbf{r}, \omega) + \theta[-\sigma(\mathbf{r}, \omega)] [-\sigma(\mathbf{r}, \omega)]^{\frac{1}{2}} \hat{\mathbf{b}}^\dagger(\mathbf{r}, \omega) \right\} \quad (42)$$

[$\theta(x)$, unit step function] and

$$\hat{H} = \int_0^\infty d\omega \hbar\omega \int d^3r \operatorname{sgn} \sigma(\mathbf{r}, \omega) \hat{\mathbf{b}}^\dagger(\mathbf{r}, \omega) \cdot \hat{\mathbf{b}}(\mathbf{r}, \omega). \quad (43)$$

Taking into account that $\sigma(\mathbf{r}, \omega)$ can be related to the imaginary part of the dielectric permittivity according to $\sigma(\mathbf{r}, \omega) = \varepsilon_0 \omega \operatorname{Im} \varepsilon(\mathbf{r}, \omega)$, we find that Eq. (42) is nothing but the equation suggested in Ref. [2] for the case of isotropic, locally and linearly responding dielectrics that also allow for amplification. Note that, as already mentioned for the more general case of Eq. (34), both the term associated with $\hat{\mathbf{b}}(\mathbf{r}, \omega)$ and the term associated with $\hat{\mathbf{b}}^\dagger(\mathbf{r}, \omega)$ in Eq. (42) give rise to positive-frequency parts of the noise current density.

As an illustration, let us consider the correlation functions $\langle \hat{\mathbf{E}}(\mathbf{r}) \hat{\mathbf{E}}(\mathbf{r}') \rangle$ and $\langle \hat{\mathbf{B}}(\mathbf{r}) \hat{\mathbf{B}}(\mathbf{r}') \rangle$ for the case where the system is in the vacuum state $|0\rangle$ [$\hat{b}(\alpha, \omega) |0\rangle = 0$]. As well known, they play an important role in the calculation of the Casimir force (see, e.g., Refs. [14,15]). Using Eqs. (10)–(13) together with Eq. (34), taking into account Eq. (33), and recalling the reciprocity of the Green tensor, one can show by straightforward calculation that

$$\langle \hat{\mathbf{E}}(\mathbf{r}) \hat{\mathbf{E}}(\mathbf{r}') \rangle = \frac{\hbar\mu_0^2}{\pi} \int_0^\infty d\omega \omega^3 \int d^3s \int d^3s' \mathbf{G}(\mathbf{r}, \mathbf{s}, \omega) \cdot \boldsymbol{\sigma}_{\text{av}}(\mathbf{s}, \mathbf{s}', \omega) \cdot \mathbf{G}^*(\mathbf{s}', \mathbf{r}', \omega) \quad (44)$$

and

$$\begin{aligned} \langle \hat{\mathbf{B}}(\mathbf{r}) \hat{\mathbf{B}}(\mathbf{r}') \rangle \\ = -\frac{\hbar \mu_0^2}{\pi} \int_0^\infty d\omega \omega \nabla \times \int d^3 s \int d^3 s' \mathbf{G}(\mathbf{r}, \mathbf{s}, \omega) \cdot \boldsymbol{\sigma}_{\text{av}}(\mathbf{s}, \mathbf{s}', \omega) \cdot \mathbf{G}^*(\mathbf{s}', \mathbf{r}', \omega) \times \overleftarrow{\nabla}', \end{aligned} \quad (45)$$

where the kernel $\boldsymbol{\sigma}_{\text{av}}(\mathbf{r}, \mathbf{r}', \omega)$ reads

$$\boldsymbol{\sigma}_{\text{av}}(\mathbf{r}, \mathbf{r}', \omega) = \int d\alpha |\sigma(\alpha, \omega)| \mathbf{F}(\alpha, \mathbf{r}, \omega) \mathbf{F}^*(\alpha, \mathbf{r}', \omega). \quad (46)$$

In the case of purely absorbing media, $\boldsymbol{\sigma}_{\text{av}}(\mathbf{r}, \mathbf{r}', \omega)$ is nothing but $\boldsymbol{\sigma}(\mathbf{r}, \mathbf{r}', \omega)$, and by application of the integral relation (9), Eqs. (44) and (45) reduce to the standard (zero-temperature) fluctuation–dissipation relations. In contrast, if amplification is allowed for, Eqs. (44) and (45) differ from the standard formulas by terms containing the non-vanishing difference between $\boldsymbol{\sigma}_{\text{av}}(\mathbf{r}, \mathbf{r}', \omega)$ and $\boldsymbol{\sigma}(\mathbf{r}, \mathbf{r}', \omega)$. In a recent study of the (zero-temperature) Casimir force in the presence of linearly amplifying (left-handed) material [4], the necessity of this correction has not been taken into account. In fact, in Ref. [4], a (locally responding, isotropic) amplifying magnetodielectric has been introduced, by simply using the formulas for the correlation functions that are valid for absorbing media and replacing therein the positive imaginary parts of the susceptibilities by negative ones. As we have just shown, this is wrong.

4 Range of validity

It remains to specify in more detail the conditions that must be satisfied to apply the quantization scheme. Let us first consider the question as to whether the Green tensor $\mathbf{G}(\mathbf{r}, \mathbf{r}', \omega)$ remains analytic in the upper complex ω half-plane if linear amplification is allowed for. In the case of absorbing media, it is well-known that solutions to the source-free macroscopic Maxwell equations [i.e., solutions to the homogeneous version of Eq. (6)] for real frequency ω can be ruled out because of their divergent spatial behavior. The same is obviously ‘even more’ true for frequencies in the upper complex ω half-plane. Since the existence of permissible solutions to the source-free equations is known to manifest itself mathematically in the form of singularities of the Green tensor, only the lower complex ω half-plane is a possible location of such singularities in the case of absorbing media. In contrast, for linearly amplifying media it may happen that permissible solutions to the source-free macroscopic Maxwell equations exist even if ω is chosen in the upper complex ω half-plane. In this case, singularities of the Green tensor in the upper complex ω half-plane would exist, which would invalidate the proof of the commutation relation in Eq. (14).

It is not difficult to imagine that such ‘unwanted’ solutions to the source-free macroscopic Maxwell equations could arise, e.g., if waves were allowed to propagate through an infinitely extended region of amplification. Since such regions do not exist in practice, their necessary exclusion from consideration is of no practical relevance. More importantly, the same problem of ‘infinite amplification length’ can also occur in the case of a region of amplification of finite extension, if waves can pass through the region repeatedly (due to multiple reflections) with a net gain per round-trip. Therefore, setups where an amplifying medium is part of an arrangement of bodies that act as a high- Q resonator must possibly be excluded from consideration. In such cases, the Green tensor fails to be a causal function in the sense of linear response theory. Of course, what breaks down is not really the principle of causality but the very concept of linear amplification—as fields with higher and higher energy develop, the non-linear dynamics can no longer be disregarded. In fact, there is no need for extra criteria to exclude from consideration setups that would mathematically support ‘unwanted’ solutions—such setups are already excluded implicitly by the assumption that the approximate concept of linear amplification is applicable. Hence, by this basic assumption, the Green tensor is forced to be analytic in the upper complex ω half-plane, and all the other important properties of the Green tensor

(in particular, its high- and low-frequency behavior and its decay behavior for large difference of the spatial arguments) are the same as in the case of absorbing media.

Next, let us answer the question of the existence of the inverse of the integral operator associated with $\sigma(\mathbf{r}, \mathbf{r}', \omega)$ in the case of linear amplification, i. e., the question of the existence of the kernel $\rho(\mathbf{r}, \mathbf{r}', \omega)$ according to Eq. (18). Obviously, $\rho(\mathbf{r}, \mathbf{r}', \omega)$ would fail to exist if eigenvalues of the integral operator associated with $\sigma(\mathbf{r}, \mathbf{r}', \omega)$ were exactly equal to zero. Since amplification is limited to a certain frequency range, and since frequency is a continuous variable, each eigenvalue $\sigma(\alpha, \omega)$ on the negative side of the eigenvalue spectrum can be made to move to the positive side by tuning the frequency, so that zero eigenvalues are possible. However, it should be emphasized that in the case of a continuous spectrum this problem is in fact harmless. It is generally true that zero occurring as a continuum eigenvalue does not really preclude the inversion of (Hermitian) operators. A familiar example is provided by the free-particle Schrödinger equation considered in the whole space. There, the continuum of δ -normalizable (plane-wave) eigenfunctions includes the (spatially constant) zero-energy eigenfunction, but this does not imply the nonexistence of the inverse of the free-particle Hamiltonian but merely the unboundedness of the inverse operator. Although there are good reasons to believe that in practice the eigenvalues $\sigma(\alpha, \omega)$ form a continuous spectrum, it seems advisable to have also a method in store to handle discrete zero eigenvalues, recalling that the calculation of physical quantities we are dealing with generally involves space and frequency integrations. As long as such a quantity remains meaningful as a whole, it does not really matter if $\rho(\mathbf{r}, \mathbf{r}', \omega)$ is not literally well-defined at individual frequencies. Thus, at least on the level of final physical results, the effect of a discrete zero eigenvalue cannot be significantly different from that of a very small but non-zero eigenvalue so that any method of regularizing $\rho(\mathbf{r}, \mathbf{r}', \omega)$ that implements this idea can be expected to give the same final results. If this is true, the problem that $\rho(\mathbf{r}, \mathbf{r}', \omega)$ might possibly fail to be literally well-defined should be irrelevant in practice. One may simply perform all calculations for a class of amplifying media for which there are no problems with $\rho(\mathbf{r}, \mathbf{r}', \omega)$. Results obtained in this way, if they are physically understandable, can then be expected to hold also without restriction.

Finally let us address the problem of the unboundedness from below of the energy eigenvalue spectrum in the case of linear amplification. Because of the lack of a lower bound, the system could evolve into states of lower and lower energy by the creation of quanta in the frequency interval where $\text{sgn } \sigma(\alpha, \omega) = -1$, in which case the theory would gradually become unrealistic. If the system described by the Hamiltonian in Eq. (35) is coupled to a second system (e. g., an atom), another aspect of the problem is that the second system might (but need not) become more and more excited, which of course also becomes unrealistic at some stage. However, such catastrophes could only occur if the theory were used beyond its range of validity. In fact, they would indicate nothing but the breakdown of the concept of linear amplification. As long as the concept of linear amplification applies, the unboundedness from below of the energy eigenvalue spectrum may be regarded as being a purely formal drawback rather than a real one. Needless to say that this unboundedness prevents one from constructing the canonical density operator—the system cannot thermalize. Therefore, it should also be clear that electromagnetic-field correlation functions and fluctuation–dissipation relations in the familiar form known for absorbing media are not applicable to amplifying media, not even at zero temperature, i. e., to the vacuum state. Nevertheless, all the correlation functions required can be calculated straightforwardly for any well-defined quantum state, in particular, for the vacuum state, as has been demonstrated at the end of Sec. 3 for the particular correlation functions needed for studying the Casimir force in the presence of linearly pumped media.

5 Summary and conclusion

We have shown that and how the very general quantization scheme developed in Ref. [1] for the macroscopic electromagnetic field in arbitrary linearly responding media in equilibrium can be extended in a rather natural way to include media which (in some frequency interval and some spatial region) are weakly pumped so that the effect of pumping can be approximately

described within linear response theory. In this sense, we have described the medium by a complex conductivity tensor $\mathbf{Q}(\mathbf{r}, \mathbf{r}', \omega)$ and allowed for negative eigenvalues of the integral operator corresponding to its real part $\boldsymbol{\sigma}(\mathbf{r}, \mathbf{r}', \omega)$. Therefore, the basic condition to apply the theory is the validity of the concept of linear amplification for the respective problem under study. It has turned out that when considering amplifying media more care and prudence is needed than in the case of robust equilibrium media, which only give rise to absorption.

Making contact with earlier work, we have shown that in the special case of amplifying linear media that can be regarded as being isotropic and locally responding, the results in Refs. [2,3] can be recovered. To illustrate the theory, we have calculated the electromagnetic-field correlation functions that determine the Casimir force in the presence of linearly amplifying media. As expected, they are quite different from the well-known correlation functions in the case of absorbing media. Since the latter are wrongly used in Ref. [4] to study the effect of linearly pumped magnetodielectrics on the Casimir force, with special emphasis on left-handed metamaterials, the results presented therein are questionable.

For a deeper understanding of the macroscopic theory considered in the present paper, it would certainly be advantageous to have also available a more microscopic model of the quantized electromagnetic field interacting with linear media that also allow for amplification, especially an analog of the well-known Huttner-Barnett-type harmonic-oscillator models frequently used to study the quantized field in absorbing media [16,17]. Such a model should include a reservoir as in the case of absorbing media, but presumably also a second, ‘inverted’ reservoir capable of being prepared in a (formal) negative-temperature state. To our knowledge, Huttner-Barnett-type models that aim to incorporate amplification have unfortunately not been developed.

We acknowledge discussions with M. Fleischhauer about the Casimir force in the presence of linearly amplifying media.

A Proof of Eq. (14)

Using Eqs. (10)–(13) and recalling the reciprocity of $\mathbf{G}(\mathbf{r}, \mathbf{r}', \omega)$, we may write

$$\begin{aligned} [\hat{\mathbf{E}}(\mathbf{r}), \hat{\mathbf{B}}(\mathbf{r}')] &= -i\mu_0^2 \int_0^\infty d\omega \omega \int_0^\infty d\omega' \int d^3s \int d^3s' \\ &\quad \times \left\{ \mathbf{G}(\mathbf{r}, \mathbf{s}, \omega) \cdot [\hat{\mathbf{j}}_{\mathbf{N}}(\mathbf{s}, \omega), \hat{\mathbf{j}}_{\mathbf{N}}^\dagger(\mathbf{s}', \omega')] \cdot \mathbf{G}^*(\mathbf{s}', \mathbf{r}', \omega') \right. \\ &\quad \left. + \mathbf{G}^*(\mathbf{r}, \mathbf{s}, \omega) \cdot [\hat{\mathbf{j}}_{\mathbf{N}}^\dagger(\mathbf{s}, \omega), \hat{\mathbf{j}}_{\mathbf{N}}(\mathbf{s}', \omega')] \cdot \mathbf{G}(\mathbf{s}', \mathbf{r}', \omega') \right\} \times \overleftarrow{\nabla}'. \end{aligned} \quad (47)$$

Applying the commutation relation (15) and employing the reality and reciprocity of $\boldsymbol{\sigma}(\mathbf{r}, \mathbf{r}', \omega)$, we may carry out one of the frequency integrals to obtain

$$[\hat{\mathbf{E}}(\mathbf{r}), \hat{\mathbf{B}}(\mathbf{r}')] = \frac{2\hbar\mu_0^2}{i\pi} \text{Re} \int_0^\infty d\omega \omega^2 \int d^3s \int d^3s' \mathbf{G}(\mathbf{r}, \mathbf{s}, \omega) \cdot \boldsymbol{\sigma}(\mathbf{s}, \mathbf{s}', \omega) \cdot \mathbf{G}^*(\mathbf{s}', \mathbf{r}', \omega) \times \overleftarrow{\nabla}'. \quad (48)$$

By means of the integral relation (9), we can now evaluate the spatial integrals. We find, on recalling the relation $\mathbf{G}^*(\mathbf{r}, \mathbf{r}', \omega) = \mathbf{G}(\mathbf{r}, \mathbf{r}', -\omega^*)$,

$$[\hat{\mathbf{E}}(\mathbf{r}), \hat{\mathbf{B}}(\mathbf{r}')] = \frac{2\hbar\mu_0}{i\pi} \int_0^\infty d\omega \omega \text{Im} \mathbf{G}(\mathbf{r}, \mathbf{r}', \omega) \times \overleftarrow{\nabla}' = \frac{\hbar\mu_0}{i\pi} \left[\int_{-\infty}^\infty d\omega \omega \text{Im} \mathbf{G}(\mathbf{r}, \mathbf{r}', \omega) \right] \times \overleftarrow{\nabla}'. \quad (49)$$

As, within the framework of macroscopic QED, the presence of any medium is irrelevant at sufficiently high frequencies, the leading asymptotic behavior of $\mathbf{G}(\mathbf{r}, \mathbf{r}', \omega)$ for frequencies in

the upper ω half-plane (including the real axis) with large absolute values is given by

$$\mathbf{G}(\mathbf{r}, \mathbf{r}', \omega) \simeq -\frac{c^2}{\omega^2} \mathbf{I} \delta(\mathbf{r} - \mathbf{r}'), \quad (50)$$

just as for the Green tensor in free space. Together with the fact that the most singular term of $\mathbf{G}(\mathbf{r}, \mathbf{r}', \omega)$ at $\omega = 0$ is $-c^2 \mathbf{L}(\mathbf{r}, \mathbf{r}')/\omega^2$, where $\mathbf{L}(\mathbf{r}, \mathbf{r}')$ is a real tensor that is purely longitudinal from both sides [and thus does not contribute to $\text{Im } \mathbf{G}(\mathbf{r}, \mathbf{r}', \omega)$ for real ω], this asymptotic behavior shows that the integral in the square brackets in Eq. (49) converges. It may be evaluated by contour-integral techniques as

$$\begin{aligned} \int_{-\infty}^{\infty} d\omega \omega \text{Im } \mathbf{G}(\mathbf{r}, \mathbf{r}', \omega) &= \text{Im } \mathcal{P} \int_{-\infty}^{\infty} d\omega \omega \mathbf{G}(\mathbf{r}, \mathbf{r}', \omega) \\ &= \text{Im} \int_{\mathcal{C}} d\omega \omega \mathbf{G}(\mathbf{r}, \mathbf{r}', \omega) = \pi c^2 [\mathbf{I} \delta(\mathbf{r} - \mathbf{r}') - \mathbf{L}(\mathbf{r}, \mathbf{r}')]. \end{aligned} \quad (51)$$

Here, the principal-value (\mathcal{P}) integral has been changed to an integral over a contour \mathcal{C} that consists of an infinitely large semi-circle in the upper half-plane (traversed clockwise), plus an infinitely small semi-circle (traversed counter-clockwise) that avoids the origin in the upper half-plane. Note that the sub-leading (weaker than ω^{-2}) singular terms of $\mathbf{G}(\mathbf{r}, \mathbf{r}', \omega)$ at $\omega = 0$ do not contribute, irrespective of the actual nature of the singularity. Inserting Eq. (51) in Eq. (49) and using $\mathbf{L}(\mathbf{r}, \mathbf{r}') \times \vec{\nabla}' = 0$, we arrive at the desired Eq. (14) $[\vec{\nabla} \times \mathbf{I} \delta(\mathbf{r} - \mathbf{r}') = -\mathbf{I} \delta(\mathbf{r} - \mathbf{r}') \times \vec{\nabla}']$.

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